# Final Exam Probability Theory (WBMA046-05)
## Solutions

24 June 2022, 08:30-10:30

- This exam contains 6 pages (including this cover page) and 5 exercises.
- Write your name and student number at the top of EACH page (including this cover page).
- Your answers should be written in this booklet. Preferably, avoid handing in extra paper. If you do not have enough space to write your solution below the exercise, you can use the back of the sheet. If this is still not enough space, you can add one (or more) extra sheet(s) to the exam. In this case, indicate it clearly in the booklet, and write your name and student number on the top of that (these) sheet(s).
- Do not write on the table below.
- Sticking to the rules above is worth 10 points.
- It is absolutely NOT allowed to use calculator, phone, smartwatch, books, lecture notes or any other aids.
- Always give a short proof of your answer or a calculation to justify it, or clearly state the facts from the lecture notes you are using (unless it is stated explicitly in the question this is not needed.).

| Exercise | Points | Score |
|---|---|---|
| 1 | 30 | |
| 2 | 20 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 20 | |
| Follow the rules (name on each page...) | 10 | |
| Total | 100 | |

**Exercise 1 (a:9, b:12, c:9 pts)**

Let $U_k$, $k \in \mathbb{N}$, be a sequence of independent random variables, distributed according to the continuous uniform distribution on $[0,1]$. For every $n \in \mathbb{N}$, let the random variable $Y_n$ be defined by

$$Y_n := \max_{i=1,\dots,n} U_i.$$

a) Show that $Y_n$ has the probability density function

$$f_n(y) = ny^{n-1}\mathbb{1}_{[0,1]}(y), \qquad y \in \mathbb{R},$$

where, as usual, $\mathbb{1}_{[0,1]}(y)$ equals 1 if $y \in [0,1]$ and 0 otherwise.
*Hint*: Consider the distribution function.

b) Compute $\mathbb{E}[Y_n]$ and $\mathbb{V}\mathrm{ar}(Y_n)$ if they exist.
*Expected form of the answer:* In both cases: a rational function in the variable $n$, i.e. a fraction whose numerator and denominator are polynomials in $n$.

c) Show that $Z_n = n(1-Y_n)$ converges in distribution (for $n \to \infty$), and state the distribution in the limit.
*Hint*: You can use the limit $(1-\frac{t}{n})^n \to e^{-t}$ as $n \to \infty$, and you might want to relate this quantity to the cummulative distribution function of $Z_n$.

**Solution**:

a) $Y_n$ takes values in $[0,1]$ (following from the same property about $U_k$) and thus

$$f_n(y) = 0 \quad \text{for any } y \notin [0,1].$$

For $t \in [0,1]$, we have

$$F_n(t) = \mathbb{P}(Y_n \le t) = \mathbb{P}(\cap_{i=1}^n \{U_i \le t\}) = \prod_{i=1}^n \mathbb{P}(U_i \le t) = t^n,$$

where the second equality holds because the random variables $U_i$ are independent. Therefore,

$$f_n(y) = \frac{\partial F_n(y)}{\partial y} = ny^{n-1} \quad \text{for any } y \in [0,1].$$

Hence, we have shown that $f_n(y) = ny^{n-1}\mathbb{1}_{[0,1]}(y)$ for any $y \in \mathbb{R}$.

b)

$$\mathbb{E}Y_n = \int_{\mathbb{R}} y f_n(y)dy = \int_0^1 yny^{n-1}dy = n\int_0^1 y^n dy = \frac{n}{n+1}.$$

$$\mathbb{E}Y_n^2 = \int_{\mathbb{R}} y^2 f_n(y)dy = \int_0^1 y^2ny^{n-1}dy = n\int_0^1 y^{n+1} dy = \frac{n}{n+2}.$$

$$\mathrm{Var}Y_n = \mathbb{E}[Y_n^2] - (\mathbb{E}Y_n)^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2} = \frac{n}{(n+2)(n+1)^2}.$$

c) For any $t \ge 0$, and $n$ large enough ($n \ge t$ will be enough for our purpose), we have

$$\mathbb{P}(Z_n \ge t) = \mathbb{P}(n(1-Y_n) \ge t) = \mathbb{P}(Y_n \le 1-\tfrac{t}{n}) = \mathbb{P}(\cap_{i=1}^n\{U_i \le 1-\tfrac{t}{n}\}) = \prod_{i=1}^n \mathbb{P}(U_i \le 1-\tfrac{t}{n}) = (1-\tfrac{t}{n})^n \to e^{-t},$$

and thus $F_{Z_n}(t) \to 1-e^{-t}$ for any $t \ge 0$. We conclude that $Z_n$ converges in distribution to a random variable with exponential distribution of rate 1.

**Exercise 2 (a:10, b:10 pts)**

An insurance company believes that people can be divided into two classes: those who are accident-prone and those who are not. The company's statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas this probability decreases to 0.2 for a person who is not accident-prone.

We assume that 30 percent of the population is accident-prone.

a) Compute the probability that a new policyholder will have an accident within a year of purchasing a policy.
*Expected form of the answer:* An exact numerical value.

b) Assume now that a new policyholder has an accident within a year of purchasing a policy. Compute the probability that he or she is accident-prone.
*Expected form of the answer:* A fraction of two integer numbers.

**Solution**:

a) We shall obtain the desired probability by first conditioning upon whether or not the policyholder is accident-prone. Consider the events

$$A_1 = \{\text{the policyholder will have an accident within a year of purchasing the policy}\},$$

$$A = \{\text{the policyholder is accident-prone.}\}.$$

Hence, the desired probability is given by

$$\mathbb{P}(A_1) = \mathbb{P}(A_1 \mid A)\mathbb{P}(A) + \mathbb{P}(A_1 \mid A^c)\mathbb{P}(A^c) = (.4)(.3) + (.2)(.7) = 0.26.$$

b) The desired probability is

$$\mathbb{P}(A \mid A_1) = \frac{\mathbb{P}(A \cap A_1)}{\mathbb{P}(A_1)} = \frac{\mathbb{P}(A)\mathbb{P}(A_1|A)}{\mathbb{P}(A_1)} = \frac{(0.3)(0.4)}{0.26} = \frac{6}{13}.$$

**Exercise 3 (10 pts)**
Xavier and Yael decide to meet at a certain location. We assume that each of them independently arrives at a time uniformly distributed between 1 p.m. and 2 p.m. Compute the probability that the first to arrive has to wait longer than 10 minutes.
*Expected form of the answer:* A fraction of two integer numbers.
**Solution**: If we let $X$ and $Y$ denote, respectively, the time past 1pm that Xavier and Yael arrive, then $X$ and $Y$ are independent random variables, each of which is uniformly distributed over $(0,60)$. The desired probability, $\mathbb{P}(X+10<Y)+\mathbb{P}(Y+10<X)$, which, by symmetry, equals $2\mathbb{P}(X+10<Y)$, is obtained as follows:

$$\begin{aligned}
2\mathbb{P}(X+10<Y) &= 2\iint_{x+10<y} f_{X,Y}(x,y)dx\,dy \\
&= 2\iint_{x+10<y} f_X(x)f_Y(y)dx\,dy && \text{(independence)} \\
&= 2\int_{10}^{60}\int_0^{y-10}\left(\frac{1}{60}\right)^2 dx\,dy \\
&= \frac{2}{60^2}\int_{10}^{60}(y-10)dy \\
&= \frac{2}{60^2}\int_0^{50} zdz \\
&= \frac{2}{60^2}\left(\frac{y^2}{2}\Big|_{y=0}^{50}\right) \\
&= \frac{2}{60^2}\frac{50^2}{2} \\
&= \frac{5^2}{6^2} \\
&= \frac{25}{36}.
\end{aligned}$$

**Exercise 4 (10 pts)**
Let $X$ and $Y$ be independent Poisson random variables with respective means $\lambda_1$ and $\lambda_2$. Compute the distribution of $X+Y$.
*Hint:* You may want to use the moment generating functions of $X$ and $Y$ (which you can compute or use directly if you remember it from the lecture notes).
**Solution**: Since $X$ and $Y$ are independent, we have

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

And because they are Poisson distributed with means $\lambda_1$ and $\lambda_2$ we get

$$M_{X+Y}(t) = \exp(\lambda_1(e^t - 1))\exp(\lambda_2(e^t - 1)) = \exp((\lambda_1 + \lambda_2)(e^t - 1)).$$

Hence, $X+Y$ is Poisson distributed with mean $\lambda_1 + \lambda_2$.

**Exercise 5 (20 pts)**
Let $\beta_1 > 0$ and $\beta_2 > 0$. Assume $X_1 \sim \text{Exp}(\beta_1)$ and $X_2 \sim \text{Exp}(\beta_2)$ are independent. Set $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Compute the joint probability density function $f_{Y_1,Y_2}$ of the random vector $(Y_1, Y_2)$.
**Solution**: Let $g(x_1,x_2) = (x_1 + x_2, x_1 - x_2)$, so that we can write

$$(Y_1,Y_2) = g(X_1,X_2).$$

Since $X_1$ and $X_2$ are exponentially distributed and independent, we have $f_{X_1,X_2}(x_1,x_2) > 0$ if and only if $f_{X_1}(x_1) > 0$ and $f_{X_2}(x_2) > 0$, that is, if $x_1, x_2 > 0$. So the *domain* where the random vector $(X_1,X_2)$ takes its values is the set

$$D = \{(x_1,x_2) \in \mathbb{R}^2 : x_1,x_2 > 0\}.$$

We will now identify the range $R = \{g(x_1,x_2) : (x_1,x_2) \in D\}$. In this case, a convenient way to find it is to note that the image under $g$ of horizontal lines $\{(s,y_0) : s > 0\}$ are diagonal lines $\{(s+y_0, s-y_0) : s > 0\} = \{(y_0,-y_0) + s(1,1) : s > 0\}$. By running over all possible values of $y_0$, we see that

$$R = \{(y_1,y_2) : -y_1 < y_2 < y_1\}.$$

Now that we have found $D$ and $R$ so that the function $g\colon D \to R$ is one-to-one. We find that $h = g^{-1}$ is given by

$$h(y_1,y_2) = \left(\tfrac{1}{2}(y_1+y_2), \tfrac{1}{2}(y_1-y_2)\right).$$

Therefore, for $(y_1,y_2) \in R$, we have

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(g^{-1}(y_1,y_2)) \cdot |J(y_1,y_2)|,$$

where

$$|J(y_1,y_2)| = \left|\det\begin{pmatrix} \partial h_1/\partial y_1 & \partial h_1/\partial y_2 \\ \partial h_2/\partial y_1 & \partial h_2/\partial y_2 \end{pmatrix}\right| = |-\frac{1}{2}| = \frac{1}{2}.$$

Thus, for $(y_1,y_2) \in R$,

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(h(y_1,y_2)) \cdot \frac{1}{2} \\ &= \frac{1}{2} \cdot f_{X_1}(h_1(y_1,y_2)) \cdot f_{X_2}(h_2(y_1,y_2)), && \text{since } X_1 \text{ and } X_2 \text{ are independent,} \\ &= \frac{1}{2} \cdot \beta_1 \cdot e^{-h_1(y_1,y_2)\beta_1} \cdot \beta_2 \cdot e^{-h_2(y_1,y_2)\beta_2}, && \text{since } X_1 \text{ and } X_2 \text{ are exponentially distributed,} \\ &= \frac{\beta_1\beta_2}{2} e^{-\frac{(y_1+y_2)\beta_1}{2} - \frac{(y_1-y_2)\beta_2}{2}}. \end{aligned}$$

We conclude that, for any $(y_1,y_2) \in \mathbb{R}^2$,

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{\beta_1\beta_2}{2} e^{-\frac{(y_1+y_2)\beta_1}{2} - \frac{(y_1-y_2)\beta_2}{2}} \mathbf{1}(-y_1 < y_2 < y_1).$$